2008

Foundations of geometry

Lawrence Michael Clarke

Foundations of Geometry

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

in

Mathematics

by

Lawrence Michael Clarke

March 2008

Foundations of Geometry

A Thesis

Presented to the

Faculty of

California State University,

San Bernardino

---

by

Lawrence Michael Clarke

March 2008

Approved by:

[redacted]

Shawnee McMurran, Committee Chair — 3/7/08 Date

[redacted]

John Sarli, Committee Member

[redacted]

Susan Addington, Committee Member

[redacted]

Peter Williams, Chair,
Department of Mathematics

Joseph Chavez
Graduate Coordinator,
Department of Mathematics

Abstract

In this paper, a brief introduction to the history, and development, of Euclidean Geometry will be followed by a biographical background of David Hilbert, highlighting significant events in his educational and professional life. In an attempt to add rigor to the presentation of Geometry, Hilbert defined concepts and presented five groups of axioms that were mutually independent yet compatible, including introducing axioms of congruence in order to present displacement. He originally presented this new material through a series of lectures during 1898-1899. Using this new presentation of geometry, he demonstrated an algebra of segments in accordance with the laws of arithmetic and showed that this was all possible without the axiom of continuity. In addition, Hilbert created new geometries in order to study Desargues's Theorem as it relates to these axioms.

ACKNOWLEDGEMENTS

First, and foremost, I would like to thank God for the many people He put in my path to help me throughout the Master's program, from classmates, professors and staff at the university to friends and associates that encouraged me along the way.

Secondly, I would like to thank my wife and children for the incredibly patient support and understanding that they provided, knowing that the work that I was doing towards my Master's degree was going to impinge on my commitments to them for a finite period of time.

Finally, I would like to thank my committee members, each of which were very responsive and helpful throughout the lengthy process of this thesis paper, from the research and writing all the way through to the culmination of the final project.

Table of Contents

List of Figures

Chapter 1

Classical Geometry: Euclid

Geometry is an ancient subject that has been around over 4000 years. Its roots are found in one form or another in nearly every human culture. The subject as we know it began in Mesopotamia, Egypt, India, and China. The Egyptians were most likely motivated by the annual flooding of the Nile, and its effect on farming. In their culture, Geometry was all about application, and often the numbers were arrived at through experimentation, observation, and trial and error. It appears that the work in the other aforementioned cultures actually was more advanced than the Egyptians. However, the ancient Greeks (800 B.C. to 150 B.C.) gave credit for the development of Geometry to the Egyptians that pre-dated them. The Greeks brought in a level of abstraction, logical deduction, and proof that had never existed before. This seems to have begun with Thales of Miletus around 600 B.C. and culminated in the works of Euclid 300 years later. Although it was widely known that Euclid had drawn from all his predecessors, nothing has been found from the earlier sources to support the level of rigor in the Math that was presented by Euclid around 300 B.C. He taught the essential features of Math in a much more general sense than had been done before. He laid the axiomatic foundation of theory, setting the standard that was used for over two thousand years. It was from his presentation of Geometry that the world learned how abstraction works. He defined terms, classified objects, and enforced a deductive presentation of theory. He presented material in such a way that a person with minimal mathematical training could benefit. And yet, he was also able to convey topics that are visible only to "the eyes of the mind," as Plato would say. He did this in his presentation of irrational and incommensurable magnitudes. His book quickly became accepted as foundational. In fact, by the time of Archimedes, who was born in 287 B.C. (and may have been a student of Euclid's), the book is often referred to and used as the basic textbook of choice. The book was second in popularity only to the Bible. Even two thousand years later, in 1783, the philosopher Immanuel Kant wrote an introduction to his philosophy that included "If you want to know what Mathematics is, you just look at Euclid's *Elements*." Math and Geometry were essentially considered synonymous.

As advanced, and widely accepted as it was, Euclid's *Elements* was not perfect. When comparing it to modern Geometry, the most notable difference is the absence of ordering - a definition of betweenness. Despite the importance that Euclid placed on definitions, he left this out. He dealt with the topic intuitively, which is how he often dealt with topics that he considered to be obvious or logical. The idea of stating axioms was very new in Euclid's time. In fact, it may, or may not, have been done in Plato's time (about 380 B.C.). Also, it was not as rigorous as modern mathematicians would like. For example, to prove the SAS theorem for congruent triangles, he placed one on top of the other (known as Euclid's superposition). However, he never stated that the line that connects two points is unique, which is required in that approach.

Over the centuries, many mathematicians worked with the *Elements*. This encouraged even greater levels of logic and rigor. At first, the focus was on the fifth postulate (on parallel lines). Since it was much longer, and more complicated, than the other statements, many attempted to prove that it was a direct result of the other four postulates. Failing that, they wanted to, at the very least, simplify it. One of the byproducts of this work was the realization that the presentation of the *Elements* was not as clear, precise, and rigorous as it could be. That is why Hilbert decided to reorganize, and restate, the tenets of Geometry. Hilbert wanted to axiomatize, formalize, and give rigorous, formal, presentations. Hilbert's approach (trying to formalize things and define them carefully and then follow through on what you can do in each field) was very influential in the first half of the 20th century.

The period that focused on perfecting the presentation of Euclidean Geometry culminated in 1899 with the work of Hilbert. He cut the last strings that tied Geometry to intuition. Formalism won a victory. And, although others tried the same thing, Hilbert's reorganization of Geometry is better known than any other. Geometry was used to teach axiomatic structure. It has only been in recent times that other areas of Math have been developed to the point that one could teach that structure outside Geometry.

Chapter 2

David Hilbert's Beginnings

David Hilbert was born January 23, 1862 at 1:00 PM to Otto and Maria Hilbert in Wehlau, near Konigsberg, the capital of East Prussia. He was their only son. He had a sister, Elise, who was six years younger than he.

Konigsberg's first connection to the world of Mathematics was due to its architecture. Seven great bridges, each with its own distinct and cherished personality, joined the banks of the nearby river, Pregel with a small island called the Kneiphof (or beer garden). The people wondered whether it was possible to walk around the city in such a way that one could cross each bridge exactly once. In 1735, Euler proved its impossibility in his published paper regarding the solution, Solutio problematis ad geometrian situs pertinentis (The solution of a problem relating to the geometry of position). It is widely considered the beginnings of Topology.

Konigsberg was also known as the birthplace and grave of its greatest son, philosopher Immanuel Kant. Since the anniversary of Kant's birth involved much ceremony, David must have been familiar with the words on the philosophers crypt:"The greatest wonders are the starry heavens above me and the moral law within me."

David did not begin school until age eight, two years later than usual, and was not very impressive in his work. David remembers himself as being dull and silly. This may have been an exaggeration. A friend later commented that "one always felt his intense...desire for the truth." The school chosen for David, Friedrichskolleg, though widely considered the best in Konigsberg (Kant was one of the graduates), was an unfortunate one for young Hilbert. Language Arts (Latin and Greek) was the focus. Mathematics came in a poor second, and science wasn't even offered. During David's youth, there was quite a collection of young, talented scientists, but these students attended Altstadt Gymnasium (schools were called gymnasiums for the mental gymnastics). David did not interact with them. One such precocious youth was Hermann Minkowski (a Russian Jew who immigrated to escape persecution).

The days at Friedrichskolleg were always remembered as unhappy ones. He focused on getting by in Latin and Greek, with his mother writing his essays for him. He knew that his future was in Mathematics, which came easily to him. There was no memorization necessary, and he could explain the problems to his teachers. At the beginning of his last gymnasium year (what we would consider the senior year in high school), he transferred to the state school, Wilhelm Gymnasium. Wilhelm placed considerably more emphasis on Mathematics, even treating some of the new developments in Geometry. By this time, Hermann Minkowski, though two years Hilbert's junior, was finishing his education. Hermann completed the eight years of education in five and one-half years and entered a local university. Meanwhile, the teachers at Wilhelm noted Hilbert's "serious scientific...and...penetrating understanding." This was the earliest glimpse of the mathematician.

In the autumn of 1880, Hilbert enrolled at the local university, at Konigsberg, and found tremendous educational freedom. Faculty members chose the classes they wanted to teach and the students chose what they wanted to learn. There were no minimum requirements, no minimum number of units, no roll call and no exams until the completing of a degree. With so much freedom, many first year students spent their time in fraternal organizations, drinking and dueling. Hilbert, however, was intoxicated with the freedom to concentrate on Mathematics. Although his father (a Prussian judge) disapproved, David enrolled in Mathematics, not law. He was entering the field of Mathematics at a time when the general atmosphere was self-congratulatory because it was felt that the rigor of Mathematics had finally reached a level that could not be surpassed. In his first semester, Hilbert heard lectures on integral calculus, determinant theory, and curvature of surfaces. Before the second semester, following the popular customs of moving from university to university, he set out for Heidelberg. He attended lectures by a professor notorious for not preparing for class. He produced on the spot what he was to say. It allowed the students the "opportunity of seeing a mathematical mind of the highest order actually in operation," as one of the students wrote. The following semester, Hilbert could have moved on to Berlin, but instead returned to the University of Konigsberg. Despite the fact that there was only one full professor of Mathematics there, David stayed there for the remainder of his eight university semesters. During that time, Minkowski returned after three semesters in Berlin. This was a time when Minkowski, at the age of eighteen, was winning awards and recognition. Hilbert took full advantage of the situation by befriending the shy, gifted Minkowski despite his father's disapproval (Minkowski was Jewish). This relationship, over the years, proved to be a great asset to each of them. It was at this point in his life, that Hilbert began spending a lot of time discussing Mathematics with other Mathematicians. Regularly, he would go for walks with Hermann Minkowski, Adolf Hurwitz, or other gifted Mathematicians. Hilbert would also go out of his way to visit accomplished professors to pick their brains, whether he agreed with their theories/philosophies or not.

When he was 23, after completing his doctoral dissertation (from the theory of Algebraic invariants), Hilbert took, and passed, the test necessary to teach at the Gymnasium (high school level). This was a common practice; it provided security if one could not get a position as a professor, since there were so few openings in any given period. He continued to meet with Mathematicians of the time, including Felix Klein, Henri Poincare, Adolf Hurwitz, and Leopold Kronecker.

He didn't seem to focus on any one field as a docent (unpaid university lecturer/teacher that is dependent upon fees paid by students). He was free to choose his topics and, contrary to what many did, chose to not repeat topics. He decided, by doing this, he could continue to educate himself, as well as his students. In fact, he set a goal of "a systematic exploration" of Mathematics. In his first few semesters, although he lectured on such varied topics as invariant theory, determinants, hydrodynamics, spherical harmonics and numerical equations, his published works were entirely in the field of algebraic invariants. After this, in March 1888, he set out on a trip to visit with twenty one prominent mathematicians. His first stop was with Paul Gordan, an expert in the field of invariants. An invariant is something that is left unchanged by some class of functions. In particular, invariant theory studied quantities which were associated with polynomial equations and which were left invariant under transformations of the variables. For example, the discriminant,

$$b^2 - 4ac,$$

is an invariant of the quadratic form,

$$ax^2 + bxy + cy^2.$$

Twenty years earlier, Gordan had produced a general form to a problem involving invariants that earned him the reputation as the "king of invariants." He had proven the existence of a finite basis for the binary forms, the simplest of all algebraic forms. In the twenty years following, none of the many European mathematicians were able to extend the proof beyond the binary forms. Certain cases were known to be true, but that is all. The problem became known as Gordan's Problem. Just before Hilbert's visit, Gordan published the second part of his "Lectures on Invariant Theory," to expound on the earlier work. Hilbert had been familiar with Gordan's Problem, but as he listened to Gordan, himself, the problem captured his imagination with a completeness that he never before experienced. He would later list the characteristics of a fruitful mathematical problem, and this met all the requirements. It was clear and easy to comprehend (this quality "attracts, the complicated repels"). It was difficult ("in order to entice us"), yet not completely inaccessible ("lest it mock our efforts"). Finally, it was significant ("guidepost on the torturous paths to hidden truths"). The problem would not let him go.

Before he even finished his travels, he had simplified Gordan's proof. It did not take Hilbert long to realize that the way every other mathematician had approached the problem (through elaborate algorithmic apparatus, just like Gordan did) was incredibly difficult with many variables and a complicated transformation group. He realized that the only way to solve the problem was to approach it from an entirely different manner. He rephrased the question as follows: "If an infinite system of forms be given, containing a finite number of variables, under what conditions does a finite set of forms exist, in terms of which all the others are expressible as linear combinations with rational integral functions of the same variables for coefficients?" His conclusion was that such a set of forms always exists.

Hilbert based his answer on a lemma involving the existence of a finite basis of a module. He developed the idea from studying Kronecker's work. The lemma was simple and almost trivial, but Gordan's general theorem followed as a direct result of it. It seemed so utterly simple that it took awhile for the mathematical world to accept that it had, indeed, been solved. Gordan, especially, fought acceptance. It took him two years to give Hilbert credit for the important offering to the field. David Hilbert was just then beginning to get attention for the mathematician that he was, as he continued to produce work in various fields of mathematics (applied and otherwise).

Chapter 3

Introduction to The Foundations of Geometry

The Foundations of Geometry is a translation of the lectures given by David Hilbert during the winter semester of 1898-1899. Hilbert started his discussion with, what we call, points, straight lines, and planes. We could, just as easily, call them something else. He, himself, summarized his foundations of Geometry: "One must be able to say at all times - instead of points, straight lines, and planes - tables, beer mugs, and chairs." The essence of Geometry is to define the relationship between these elements. This was Hilberts attempt to choose for Geometry a complete set of mutually independent, yet compatible, axioms, and to deduce, from these, the most important Geometrical theorems in a clear and simple manner. He built his foundations on twenty-one axioms. Chapter I organizes the axioms into five groups: connection, order, parallels, congruence, and continuity.

Chapter 4

The Five Groups of Axioms

The seven axioms in the first group establish a connection between the elements of space (the points, lines, and planes) mentioned above.

I-1 Two distinct points determine a line.

I-2 Any two distinct points of a line determine that line, so a line can be called by any two points on the line and yet still be the same line.

I-3 Three non-collinear points determine a plane.

I-4 Three non-collinear points of a plane completely determine that plane.

I-5 If two points of a straight line lie in a plane, then every point of that line lies in the plane.

I-6 If two planes have a point in common, then they have at least a second point in common.

I-7 On every straight line, there exists at least two points, in every plane, at least three non-collinear points, and in space, at least four non-coplanar points.

From these axioms, two theorems were presented that detailed the results of any intersection of the elements above. Among the theorems, it was stated that two straight lines within a plane intersect in, at most, one point.

The second group, the "axioms of order," deals with the idea of betweenness and sequential order.

II-1 If A, B, and C are points on a straight line and B is between A and C, it is also between C and A.

II-2 If A and C are two points of a straight line, then there exists at least one point B lying between them, and at least one point D such that C lies between A and D.

II-3 Of any three points on a given line, exactly one lies between the other two.

II-4 Any four points of a straight line can be arranged such that B lies between A and C and also between A and D. C also lies between A and D and B and D.

II-5 If A, B, and C are three points not lying on a straight line, and line a lies in the same plane and intersects segment AB (between A and B), it will also intersect segment BC or AC (see Figure 4.1).

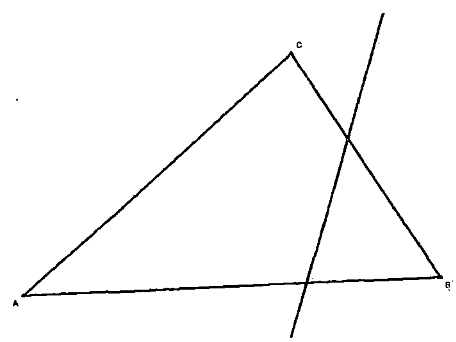

Figure 4.1: Axiom II-5: Intersection Between Line and Sides of Triangle

Additional theorems were presented as a result of the first two sets of axioms. Included among these is the idea that there are an infinite number of points between any two points on a line. Also, a line (or a polygon) divides a plane into two regions, as a plane does the same to space.

The third group of axioms consist of a single axiom: the axiom of parallels.

III In a plane, through any point A (not lying on line a) exactly one line can be drawn that does not intersect the line. This new line is said to be parallel to line a.

The fourth group, consisting of six axioms, defines the idea of congruence, or displacement. The first three concern line segments, the next two deal with angles, and the sixth combines the two concepts.

IV-1 If A and B are two points on a line, and if A' is a point on that or another line, one (and only one) point on a particular side of A' can be found, B', such that the segment lengths are congruent. In other words, the distance from A to B is identical to the distance from A' to B'. Also, every segment is congruent to itself.

IV-2 Segment congruence is transitive. Therefore, if AB is congruent to $A'B'$, which, in turn, is congruent to $A''B''$, then AB is congruent to $A''B''$.

IV-3 Segments can be added. If A, B, and C are three collinear points (AB and BC having only the B point in common) and A', B', and C' have the same relation on the same or another line, and AB is congruent to $A'B'$ and BC is congruent is $B'C'$, then AC is congruent to $A'C'$.

IV-4 An angle, created with rays OH and OK intersecting at vertex O , is congruent to itself, and can be written as $\angle HOK$ or $\angle KOH$.

IV-5 If an angle is congruent to two other angles, the other angles are congruent to each other.

IV-6 If two sides, and the included angle, of two triangles are congruent, then the remaining interior angles of the two triangles are also congruent.

Direct results of these axioms include both the SAS and ASA theorems. The first states that if two sides, and their included angle, are congruent to the corresponding parts of another triangle, the two shapes are congruent. The second states that if two angles, and their included side, are congruent to the corresponding parts in another angle, then the two polygons are congruent.

In addition to these standard proofs, a number of other theorems are shown to be a result of these axioms. One such interesting byproduct is the proof that if two angles are equal in measure, their corresponding supplementary angles are also congruent. This is proven by using the original angles and building triangles around them in such a way that the vertex of each of those angles is located on a side of the triangle and the rays of the angles meet the opposite vertices within the new triangle. This creates two new pairs of congruent triangles within the originals. Since the corresponding parts of congruent triangles are, themselves, congruent, our point is proven. Therefore, supplementary angles of congruent angles are congruent to each other.

A list of other results include:

- When congruent angles are bisected, their corresponding half-angles are also equal in measure.
- When a pair of half-angles are congruent to another pair of half angles, the resultant angles (from which the half-angles were created) are also congruent.
- All right angles are congruent to each other.
- If all three sides of a triangle are congruent to the sides of another triangle, the triangles themselves are congruent.
- If there are two congruent n-sided plane figures, and P is a point on the plane of the first shape, P' can be found in the second figure such that the new shapes, (A, B, C, ..., P) and (A', B', C', ..., P') are congruent. If the original shapes include at least three non-collinear points, P' is a unique point.
- In the previous note, if P is any point (not necessarily on the plane, P' is still unique if the original shape contains at least four non-collinear points.
- If two parallel lines are cut by a third line, a transversal, the alternate-interior angles are congruent, as are the exterior-interior angles. Conversely, if two lines are cut by a third and the previously mentioned angles are equal in measure, the two lines are parallel.
- The angles of a triangle add to two right angles.

The fifth group of axioms, that of continuity, consists of a single statement, the axiom of Archimedes. It states that beyond any stated point on a line is another. Therefore, the points on a line are infinite in nature. It is a linear axiom of continuity.

To these five sets of axioms, Hilbert added a final assertion. He stated that it is impossible to create a new geometry by adding other elements (to a system of points, straight lines, and planes) and still hold these given sets of axioms as valid. Since this geometry cannot be extended, it is complete.

Chapter 5

Compatibility and Mutual Independence of the Axioms

It is not possible to create a geometry from these five sets of axioms that, in turn, contradicts some of these axioms. They do not contradict each other. In fact, Hilbert used these axioms to create a geometry that consisted of algebraic numbers created from five operations: addition, subtraction, multiplication, division, and $\sqrt{1+\Omega^2}$, where Ω represents a number already included in the set of numbers created. A pair of such numbers (x, y) represent a point in this geometry, and a ratio of three such numbers $(u : v : w)$ represents a straight line. It was similar to our standard notation of a linear equation. We would write it as $ux + vy + w = 0$.

The members of the domain, Ω, are real numbers. We can arrange these numbers by magnitude. We must only have the x values or the y values on this line increasing or decreasing to give an order to them. This would satisfy many of the axioms from the first three sets (connection, order and parallels). For instance, in order to demonstrate axiom five of the second set, we just assume that all points (x, y) that make $ux + vy + w$ more than, or less than, zero fall on opposite sides of the line $ux + vy + w = 0$ (see figure 5.1).

The line that goes through the point C is represented by the equation. Since it is given that the line does not go through any of the vertices of the triangle, it must go through one of the opposite sides. This convention does not conflict with any of the others that we have presented.

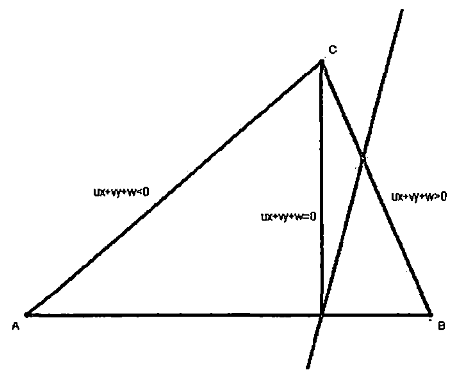

Figure 5.1: Illustration of the Meaning of Axiom II-5

Translations of segments and angles are defined in the usual way, with x=x+a and y=y+b producing such. In addition, a rotation of an angle can be produced by naming the vertex $(0,0)$ and one line as the initial segment. We will name a point on it $(1,0)$. When the angle is rotated, the point (x,y) becomes (x',y'), where

$$x' = \frac{a}{\sqrt{a^2+b^2}}x - \frac{b}{\sqrt{a^2+b^2}}y$$

and

$$y' = \frac{b}{\sqrt{a^2+b^2}}x + \frac{a}{\sqrt{a^2+b^2}}y.$$

Since

$$\sqrt{a^2+b^2} = a\sqrt{1+(\frac{b}{a})^2}$$

belongs to the domain, the axioms of congruence also hold. The same is true for the axiom of Archimedes, which is the linear axiom of continuity. If this geometry had been presented with a domain of all real numbers, all five sets of axioms still would have been valid. For this demonstration, we instead use a countable set.

Now that we have demonstrated that the axioms above are not contradictory, we need to show that they are mutually independent; None of them can be deduced from the others.

First of all, it is easy to see that the axioms of connection, order and congruence (groups I, II, and IV, respectively) can not be derived from the related axioms of their own group. In fact, the first two groups were used to form a basis for the other groups.

We must, then, show that each of the last three groups (axioms of parallel, congruence, and continuity) is independent of the others.

The first statement of parallels states that given a line, a, and a point A not on that line, there is a singular co-planar line that is parallel to a and intersects A. This can be demonstrated through the use of the axioms of connection, order and congruence.

To illustrate this, connect point A to a point B on line a (see figure 5.2).

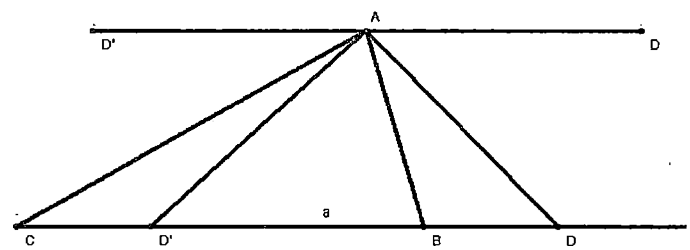

Figure 5.2: Axiom III: Statement of Parallels

Locate another point on line a and call it C. Complete the triangle ABC in the same plane as the line a and the points A, B and C. The straight line through A can not intersect with line a. If it did, say in point D (with B between C and D), then we can find the point D', such that $AD \equiv BD'$. As a result of the congruence of the two sides, together with the sharing of side AB and the congruence of the included angles, the two triangles, ABD and BAD', are congruent. Therefore, the angles ABD and BAD' are equal in measure. Since angles ABD' and ABD are supplementary (together they form the line $D'D$), angles BAD and BAD' must also be supplementary (same logic). This, however, can not be true, since this would require the two lines to meet in more than one point, which contradicts one of the earliest theorems stated from the first group of axioms (that of connection), which states that two lines meet in no more that one point. Therefore, by contradiction, we have proven that the lines are parallel.

The only remaining part of this statement to demonstrate is that this line is singular in nature - it is the only line through A that is parallel to a. To do this, Hilbert used the elements of Ω to represent the points in his geometry. These elements are then restricted to the interior of a sphere. The congruences are defined to be transformations of the sphere onto itself. This non-Euclidean geometry obeys all the axioms except for the axiom of Euclid (the statement of parallels). Since the parallel nature has already been shown in ordinary geometry, that of non-Euclidean is an immediate consequence.

Chapter 6

Independence of the Axioms of Congruence and Continuity

We shall now demonstrate the independence of the axioms of congruence by demonstrating that axiom IV, 6 ($SAS \Rightarrow$ equiangular triangles) can not be deduced from the remaining axioms by any logical reasoning.

Define points, straight lines, angles, and planes as one would in ordinary Euclidean geometry. Let the two points A_1 and A_2 have co-ordinates of (x_1, y_1, z_1) and (x_2, y_2, z_2), respectively. We will define the length of the segment A_1A_2 as the positive value of the expression

$$\sqrt{(x_1 - x_2 + y_1 - y_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Furthermore, two segments will be called congruent if, and only if, they have equal lengths in the sense just defined. In this geometry, of all the five groups of axioms, only part three of axiom four need be proven. The rest are immediately evident. Axiom IV, part 3, presented the addition of segments and the fact that the sums of congruent segments are also equal.

Let x, y, and z be three points on a straight line. Let the following be the parametric representations of these points using t as the parameter.

$$x = \lambda t + \lambda'$$

$$y = \mu t + \mu'$$

$$z = vt + v'$$

The Greek letters all represent constants. These substitutions allow representation of affine geometry. If t_1 , t_2 , and t_3 (decreasing values of the parameter) correspond to points A_1, A_2, and A_3 , we can plug these values into the formula for distance to determine the lengths of the segments between the points. As a result, we show the following lengths:

$$A_1A_2 = (t_1 - t_2)\left|\sqrt{(\lambda+\mu)^2 + \mu^2 + v^2}\right|$$

$$A_2A_3 = (t_2 - t_3)\left|\sqrt{(\lambda+\mu)^2 + \mu^2 + v^2}\right|$$

$$A_1A_3 = (t_1 - t_3)\left|\sqrt{(\lambda+\mu)^2 + \mu^2 + v^2}\right|$$

So, as can be seen, the length of the first two equal the third, as it should. This fulfills Axiom IV-3, which states that segments can be added. However, Axiom IV-6 (SAS), or at least the first theorem of congruence for triangles, does not always hold. As an example, we will use the $z = 0$ plane, with O at $(0, 0)$, A at $(1, 0)$ and B at $(0, 1)$. C will be at the midpoint of segment AB, at $(\frac{1}{2}, \frac{1}{2})$.

In this drawing (see figure 6.1), by plugging the co-ordinates into the definition of length, we find that, as expected, the lengths of AC and CB are equal, at $\frac{1}{2}$. (See figure 6.1).

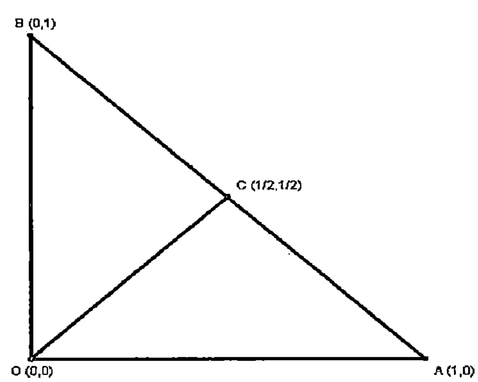

Figure 6.1: Axiom IV-3: Segments can be added

By the Side-Angle-Side Postulate (Axiom IV-6), with these sides equal in length and OC being a shared side by both right triangles OCA and OCB, we would expect that the lengths of OA and OB to be equal. We would be wrong. The length of OA is 1, but the length of OB is $\sqrt{2}$. Hilbert states that it would not be difficult to find two triangles in this geometry in which Axiom IV-6, itself, would not be valid.

In order to demonstrate the independence of the axiom of Archimedes, Hilbert produced a geometry in which all the axioms were valid except this one.

To do this, he constructed a domain, $\Omega(t)$, by adjoining t to Ω. It consists of all of the algebraic functions of t that can be obtained from the four operations and the fifth operation, $\sqrt{1+\omega^2}$, in which ω represents any function that arises from the application of these five operations. The elements of this domain produce a countable set of real numbers. The domain contains only real functions of t.

Let c be any function of this domain. Since c is an algebraic function of t, it can not vanish for more than a finite number of values of t. Also, for a sufficiently large values of t, it must remain positive or negative.

Now lets view the functions of the domain as a type of complex number. In this system of complex numbers, all the normal rules of operations hold. Also, if a and b are distinct numbers in this system, one is considered to be smaller, or greater, than the other, for sufficiently large t. And so we say, $a > b$ or $b > a$. And so, we are able to arrange the members of this system in order of magnitude. This inequality remains valid even when equal values are added to both magnitudes, or equal positive values are multiplied to each.

Since t is a variable adjoined to Ω, it is infinitely large over the algebraic numbers and is larger than any term in Ω. Therefore the inequality $n < t$ certainly holds for any term n that is an element of Ω. As a result, the expression $n - t$ is always negative for sufficiently large values of t. We express this by saying that any multiple of 1, a member of the domain Ω and a positive value, always remains smaller than t.

We can now create a geometry from the complex numbers of the domain $\Omega(t)$ in the same manner that we designed one for Ω. We will regard a system of three numbers (x, y, z) as a point and a ratio of any four numbers $(u : v : w : r)$, where they are not all zero, as a plane. Finally, the existence of the equation $xu + yv + zw + r = 0$ expresses the point (x, y, z) lies in the plane $(u : v : w : r)$. In this geometry, a straight line will be all points that lie in the same two planes. If we adopt the same conventions as we did in the geometry of Ω, we have a "non-Archimedean" geometry, since in this geometry, all of the axioms, except for that of Archimedes, are valid.

This is because we can lay off the segment 1 upon the segment t an arbitrary number of times without reaching the endpoint of the segment t, which contradicts the axiom of Archimedes, which states that if you laid if off enough times, it would surpass the original length.

Chapter 7

The Theory of Proportion

In order to provide a basis upon which to build, Hilbert pointed out that real numbers form a system that have a number of properties. These properties include:

1. Addition is defined. From a and b, c is obtained. So, $a + b = c$ or $c = a + b$.

2. The additive identity exists. There exists the definite number 0, such that, for every number a, $a + 0 = a$ and $0 + a = a$.

3. Existence and uniqueness of addend/sum relationships. For any two given numbers, a and b, there exists a unique x or y, such that $a + x = b$ or $y + a = b$.

4. Multiplication is defined. $ab = c$ or $c = ab$.

5. The multiplicative identity exists. There exists the definite number 1, such that, for every a, $a \times 1 = a$ and $1 \times a = a$.

6. Uniqueness of factor/product relationships. For any non-zero a and b there exist a unique x and y such that $ax = b$ and $ya = b$.

   In addition, for the arbitrary numbers, a, b, and c, the following operations hold.

7. Associative Property of Addition. $a + (b + c) = (a + b) + c$.

8. Commutative Property of Addition. $a + b = b + a$.

9. Associative Property of Multiplication. $a(bc){=}(ab)c$.

10. Distributive Property, part one. $a(b+c) = ab + ac$.

11. Distributive Property, part two. $(a+b)c = ac + bc$.

12. Commutative Property of Multiplication. $ab = ba$.

    In addition to these, there are statements of order.

13. For any two distinct numbers, a and b, one of these (say a) is greater than the other. The other number is said to be the smaller of the two, and the relationship is written as $a > b$ and $b < a$.

14. Transitivity of order. If $a > b$ and $b > c$, then $a > c$.

15. Addition of equal value to both sides does not alter the relationship. If $a > b$, then $a + c > b + c$ and $c + a > c + b$.

16. Multiplication of an equal positive value to both sides does not alter the relationship. If $a > b$ and $c > 0$, then $ac > bc$ and $ca > cb$.

17. The final property is the Theorem of Archimedes. This simply states that for all positive numbers a and b, it is possible to add a to itself enough times so that it totals more than b. In other words, $a + a + ... + a > b$ for a large enough number of $a's$. A system that includes only a portion of these properties is a complex number system, or simply a number system. A number system can be Archimedean or non-Archimedean, depending on whether or not the last property holds. Not every one of the properties is independent of the others. Also, it is possible to have all of the first sixteen hold, but not the last. This is true for the number system, $\Omega(t)$, that was created earlier.

We will now use as the basis of our discussion all of the plane axioms with the exception of Archimedes. With these axioms, independent of Archimedes', we will establish Euclid's theory of proportion.

To do this we will first demonstrate a special case of Pascal's theory on conic sections. We will refer to it simply as Pascal's theorem. The theorem states that given two sets of points A, B, C and A', B', C' situated on opposite sides of intersecting lines

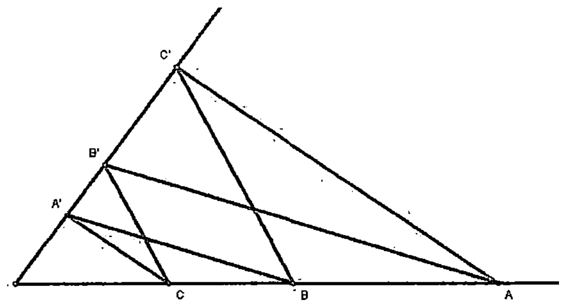

Figure 7.1: Pascal's Theory of Proportion

(but not at the intersection), the following relationship occurs. If CB' is parallel to BC' and CA' is parallel to AC', then BA' is also parallel to AB'(see figure 7.1).

In a right triangle, the base a is uniquely determined by the hypotenuse c and the base angle α. This fact, presented by Hilbert as $a = \alpha c$, is usually presented today as a trigonometric definition, $cos\alpha = \frac{adjacent}{hypotenuse}$, which corresponds with $a = c \times cos\alpha$. Therefore, αc represents a definite segment if c is given length and α is any given acute angle (see figure 7.2).

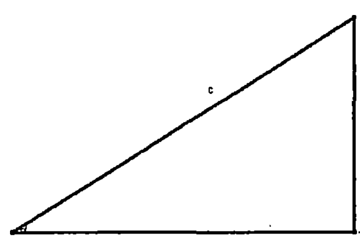

Figure 7.2: Cosine: Hilbert style

Since this is a segment, we can again multiply it by another (cosine of an) angle, say β, the other acute angle in the right triangle mentioned. In fact, we find that $\alpha\beta c \equiv \beta\alpha c$, regardless as to the measures of these two acute angles. By substituting in the appropriate sides from the definitions of cosine, it can be seen that this gives us $\frac{a}{c}b = \frac{b}{c}a$. Therefore, α and β are interchangeable.

In order to prove this statement, we will start with the segment $c = AB$ (see figure 7.3).

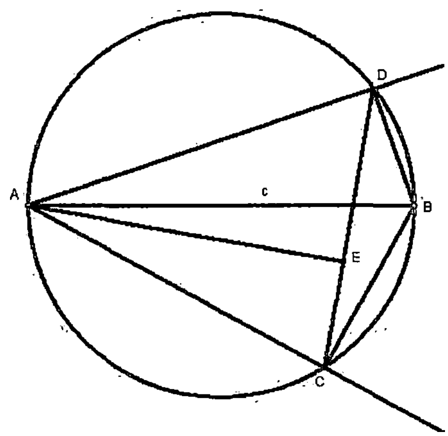

Figure 7.3: Angles are interchangeable

Using A as the vertex, we will lay off angles α and β on opposite sides of c. Then, from B, we will draw perpendicular lines, BC and BD, to the opposite sides of α and β, respectively. Next we will sketch the segment CD and the perpendicular line, AE, to it from A.

Since ACB and ADB are right angles, the points A, B, C, D are all situated on the same circle. Consequently, the angles ACD and ABD are congruent, since they inscribe the same segment in a circle. However, the measures of angles ACD and CAE add to ninety degrees, since AEC is a right angle. The same can be said of the two angles ABD and BAD (since ABD is a right angle). Since the total angle measurement of ACD and CAE (ninety degrees) is the same as the total angle measurement of ABD and BAD, and the angles ACD and ABD are equal in measure, CAE and BAD are also equal in measure. Therefore, the measure of angle CAE is β. Consequently, the measure of angle DAE is α.

From this, we are able to derive the following congruences of segments:

$\beta c = AD$, $\alpha c = AC$,$\alpha\beta c \equiv \alpha(AD) \equiv AE$, and $\beta\alpha c \equiv AE$. This, therefore, proves our earlier statement, that the two angles are interchangeable.

We will again use the figure that we used for Pascals theorem, with additional labeling. We have labeled each intersection and each segment (see figure 7.4).

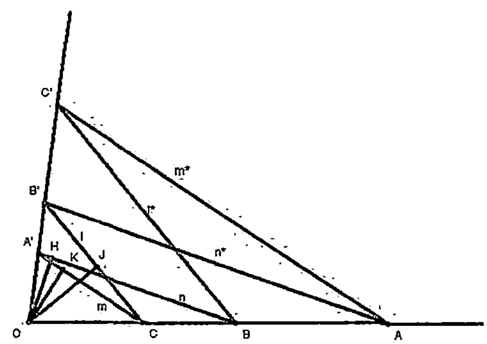

Figure 7.4: Additional Labeling of Pascal's Figure

From the vertex O, we let fall perpendiculars upon l, m and n. We denote the intersection with them L, M, and N, respectively. We will refer to the angles LOA' and LOC as λ and λ', respectively. Likewise the angles MOA', MOC, NOA', and NOC will be denoted μ',μ,v' and v, respectively.

Recall that when Hilbert refers to an angle in an equation, he is actually referring to the cosine of that angle. Therefore he is able to present them as rational equations. So, when we relate the perpendiculars in terms of their hypotenuse and base angle, we get the following congruences:

1. $\alpha B' \equiv \lambda' C$

2. $\mu A' \equiv \mu' C$

3. $v A' \equiv v' B$

   Since segments l and m are parallel to $l*$ and $m*$, respectively, their perpendiculars from vertex O must coincide. Therefore, we also have

4. $\lambda C' \equiv \lambda' B$

5. $\mu C' \equiv \mu' A$

   By multiplying both sides of congruence (3) by $\lambda'\mu$, and remembering that, as we have already seen, these symbols are commutative, we have $v\lambda'\mu A' \equiv v'\mu\lambda' B$. In this congruence, we replace $\mu A'$ on the left and $\lambda' B$ on the right by their values noted in statements (2) and (4), respectively. Our resulting congruence is $v\lambda'\mu' c \equiv$

$v'\mu\lambda c'$. Using the commutative property of these elements, we get $v\mu'\lambda'c \equiv v'\lambda\mu c'$. By use of the first and fifth congruences listed above, we can replace $\lambda'c$ and $\mu c'$ with $\lambda b'$ and $\mu'a$, respectively. This will give us $v\mu'\lambda b' \equiv v'\lambda\mu'a$, which can be manipulated to provide $\lambda\mu'vb' \equiv \lambda\mu'v'a$. Consequently, we can conclude that

6. $vb' \equiv v'a$.

   This translates to, in modern language, the cosine of $B'OH$ multiplied by b' (OB') being eqivalent to the cosine of HOA multiplied by a (OA). This can be verified by using the basic definition of cosine in triangles representing the ratio of the adjacent side to the hypotenuse. Both sides of congruence (6) produces the perpendicular that I have named OH.

From OH, drop perpendiculars to the points A and B'. These two lines coincide to form the line AB'. Since $n* = AB'$ forms a right angle with the perpendicular to n, $n*$ is parallel to n. This establishes the truth of Pascal's Theorem mentioned a few pages ago.

Using the same figure, we can demonstrate Pascal's theorem. Since the measures of angles OCA' and $OD'B$ are congruent, angle $A'CB$ is supplementary to each. Since quadrilateral $A'CBD'$ has opposite angles that are supplementary, a circle can be drawn that inscribes it (see figure 7.5).

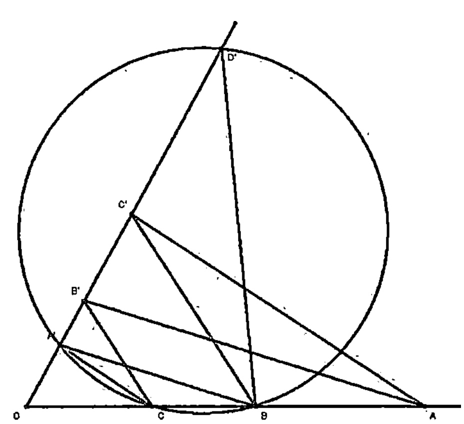

Figure 7.5: Demonstrating Pascal's Theorem

In addition, angles OBA' and $OD'C$ each have their vertices on the circle and cut the same arc from that circle. Therefore, the angles are congruent.

CA' and AC' are parallel by hypothesis, therefore the measures of angles OCA' and OAC' are equal. By combining this statement with the earlier one stating the congruence of OCA' and $OD'B$, we note that the measure of angle $OD'B$ is congruent to that of OAC'.

Now, since $BAD'C'$ is an inscribed quadrilateral, the opposite angles are supplementary. Therefore, BAD' and $D'C'B$ relate in this manner. Given that $D'C'B$ is supplementary to both OAD' and $OC'B$, the two latter angles are congruent. So, we have $OAD' \equiv OC'B$. However, since CB', by hypothesis, is parallel to BC', we also have $OB'C \equiv OC'B$.

Given the fact that the measure of angle $OC'B$ is congruent to both OAD' and $OB'C$, the latter two angles must also be congruent to each other. As a result, $CAD'B$ is also an inscribed quadrilateral and $OAB' \equiv OD'C$ (see figure 7.6). Combining this congruence with the earlier one relating OBA' to $OD'C$ shows that BA' and AB' are parallel, as Pascal's theorem requires.

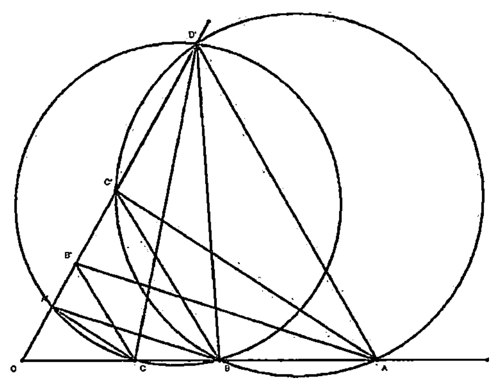

Figure 7.6: Demonstrating Pascal's Theorem II

If D' coincides with A', B', or C', it is necessary to modify the method.

Chapter 8

An Algebra of Segments based upon Pascals Theorem

Pascal's Theorem allows us to introduce into geometry a method of calculating with segments, in which all the rules for calculating with real numbers remain valid without any modification.

Since we are making a connection with algebra, we will replace congruent and $\equiv$ with equal and $=$. If we have a line segment with points A, B, and C on it, and we define $a = AB$, $b = BC$, and $c = AC$, we can show the sum of the first two segments to be the third. Therefore, $c = a + b$. Since a and b are the smaller segments, and c the larger, $a < c$, $b < c$ and $c > a$, $c > b$. From the linear axioms of congruence, we can see that both the associative and commutative laws of addition are valid.

We will now define multiplication geometrically. To do so, we will use a right-angle construction (see figure 8.1).

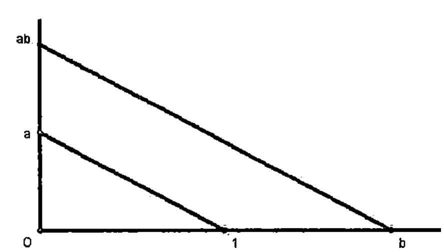

Figure 8.1: Multiplication Defined Geometrically

With the vertex named O, on one side, we will select points to represent the unit length (one) and b. On the other leg of the right angle, we will choose a point to represent a. After drawing a segment to connect one and a, we will draw a line through b and parallel to the line connecting the other two points. This parallel will cut, from the opposite side, a length of c. We will call this length the product of a and b, and indicate this by writing $c = ab$.

We will now demonstrate, that for this definition of multiplication of segments, the commutative law, $ab = ba$, holds. For this purpose, we use another diagram.

We will first construct ab, as above, using the lines l and m (see figure 8.2).

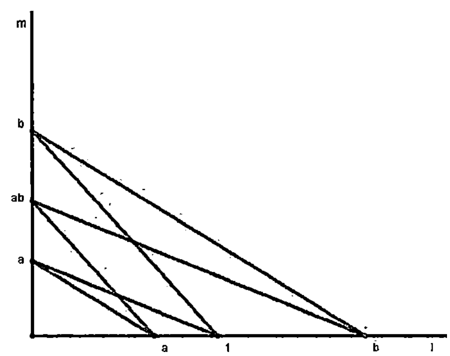

Figure 8.2: Commutative Law of Multiplication

In addition, along l, we will draw segment a on the opposite side of 1 from b, and also draw b on line m. Connect 1 with b on the opposite side. Draw a parallel to this line through the point a on l. Where it intersects with line m, we have the value ba. But, by Pascal's Theorem, used above, since the dashed lines are parallel, ba coincides with the point ab. Therefore, we have $ba = ab$.

In order to show the associative law of multiplication, $a(bc) = (ab)c$, holds for multiplication of segments, we construct $d = bc$, then da, and $e = ba$, and finally ec (see figure 8.3).

According to Pascal's Theorem, da and ec coincide. By substitution, $da = ec$, implies $(bc)a = (ba)c$. If we apply the commutative property just proven, $a(bc) = (ab)c$. Therefore, the associative property of multiplication holds.

Finally, the distributive property, $a(b + c) = ab + ac$, holds for our algebra of

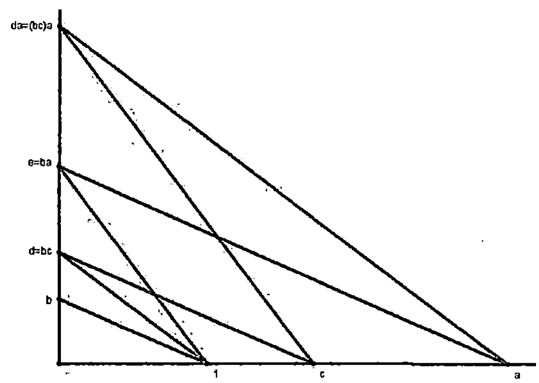

Figure 8.3: Associative Law of Multiplication

segments. In order to demonstrate this, we construct the segments ab and ac by drawing parallel lines to one that connects 1 to a (see figure 8.4).

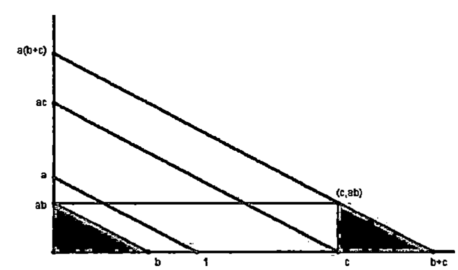

Figure 8.4: The First Distributive Law

We then mark $b+c$ on the same side as the 1 and draw another parallel in order to construct $a(b+c)$. Finally, we draw through the extremity of the segment c, a straight line parallel to the other side of the right angle. From the congruence of the two right triangles and the application of the theorem about the equality of the opposite sides of a parallelogram, we are able to determine that for any arbitrary segments b and c, there exists an a, such that $c = ab$. This segment is denoted $\frac{c}{b}$ and is called the quotient of c by b.

By use of the preceding algebra of segments, we can establish Euclid's theory of proportion free of objections and without making use of the axiom of Archimedes.

If a, b, a', b' are any segment whatever, the proportion $a : b = a' : b'$ expresses nothing else than the equation $ab' = ba'$.

Two triangles are called similar when the corresponding angles are congruent. Therefore, if a, b, and a', b' are homologous sides of two similar triangles, we have the proportion $a : b = a' : b'$.

First, we will look at the special case, when the angles between a and b and between a' and b' are both right angles. We start by placing them together, having them share a common right angle and vertex, at O (see figure 8.5).

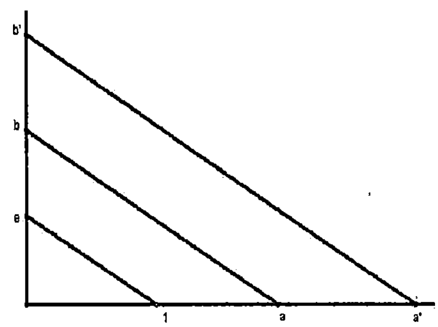

Figure 8.5: Similar Triangles

From one side of the vertex, we lay off the unit measure, 1. We then draw a line parallel to the hypotenuses of the triangles. We name the intersection with the other side e. By our definition of multiplication of segments, $b = ea$ and $b' = ea'$. By solving each of these equations for e, and setting the two amounts equal to each other, we get the proportion $\frac{b}{a} = \frac{b'}{a'}$, or $b : a = b' : a'$. Therefore, the triangles are similar.

In figure 8.6, we return to the general case. In each of the two similar triangles, find the point of intersection of the three angle bisectors. Denote these points S and S'.

From these points, let fall perpendicular segments to the three sides. Name each of these segments r and r', respectively. We will name the six segments created a, b, c, d, e, f, and a', b', c', d', e', f', respectively. This gives us proportions $\frac{a}{r} = \frac{a'}{r'}$ and $\frac{b}{r} = \frac{b'}{r'}$. By adding similar values to each side of the equation, we get $\frac{a+b}{r} = \frac{a'+b'}{r'}$. The same can be done for each of the homologous sides of the triangles. We have, therefore, proven that the proportion $a : b = a' : b'$ holds in the general case.

From this information, Hilbert stated the following theorem: If two parallel lines cut from an angle line segments a, b and a', b', respectively, then we will always

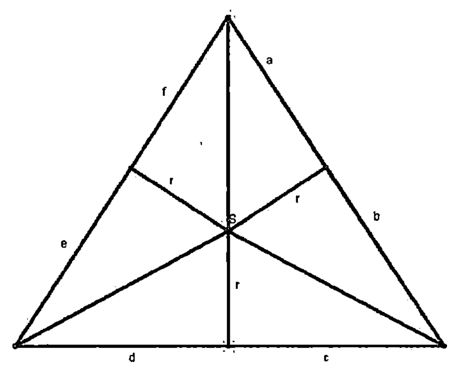

Figure 8.6: General Case

have the proportion $a : b = a' : b'$. Conversely, if the four segments fulfill this proposition and if a, a' and b, b' are laid off upon the two sides respectively of an arbitrary angle, then the straight lines joining the extremities of a and b and a' and b' are parallel to each other.

Hilbert then describes a second system of segments - a system of "negative" in contrast to the "positive" segments with which we have been working. In addition, the zero segment (consisting of a single point) is introduced. From these introductions, we are able to demonstrate all the properties put forth for his theory of proportion. Specifically, we note the multiplicative identity, the fact that this geometry of segments is an integral domain, and the transitive nature of multiplication of segments. And so we have $a \times 1 = 1 \times a = a$. Secondly, if $a \times b = 0$, then either $a = 0$ or $b = 0$. Finally, if $a > b$ and $c > 0$, then $ac > bc$.

Hilbert then unites the positive and negative set of segments into a single system that is, essentially, the Cartesian co-ordinate system that we know today. The segment lengths, x and y, represent the co-ordinates, which may be positive, negative, or zero.

Now, let l be a line that passes through O, the intersection of the two perpendicular axes, X and Y (see figure 8.7).

The line l will also pass through a point C, having co-ordinates (a, b). If another point on l, say P, has the co-ordinates (x, y), it follows from our earlier work, that $a : b = x : y$. As a result, $bx = ay$. Although we would tend to write the equation of the

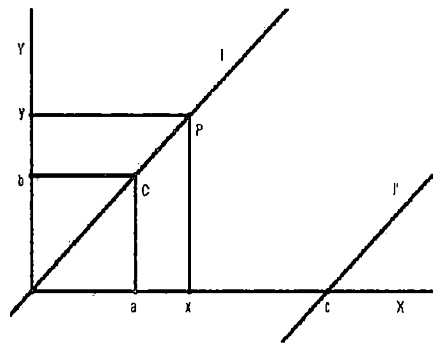

Figure 8.7: Co-ordinate System

line as $y = \frac{b}{a}x$, Hilbert wrote it as $bx - ay = 0$. If l' is a straight line parallel to l that cuts X at the segment c, we can obtain the equation of the new line by replacing x by $x - c$. This gives us $b(x - c) - ay = 0$, which is equivalent to $bx - ay - bc = 0$. Due to the fact that these equations were presented in generalities, we can conclude that, independent of the axiom of Archimedes, we can represent any straight line in a plane by a linear equation in x and y. Conversely, every linear equation represents a straight line when the co-ordinates are segments in this geometry.

So far, we have made no use of the axiom of Archimedes. If we now assume the validity of this axiom, we can relate a correspondence between the points on any straight line and the real numbers.

To demonstrate the axiom of Archimedes, Hilbert chose the numbers 0 and 1. These are represented by line segments (with 0 being a single point). He then bisected the segment n times. This produced the value, or segment length, of $\frac{1}{2^n}$. He laid this segment off m times in both the positive and negative directions from 0. This created a segment from $-\frac{m}{2^n}$ to $\frac{m}{2^n}$. Each of these points correspond to a single, definite real number. To every element of the algebraic numbers Ω, there exists a corresponding point on a straight line. However, whether there corresponds a point to every real number cannot be established in general. It depends on the geometry that is being referenced.

However, it is possible to generalize this to include irrational elements without exception. Within this larger group, all the axioms hold, and it is nothing more than the ordinary geometry of space.

Chapter 9

The Theory of Plane Areas

This new section is also based upon all the plane axioms excluding the axiom of Archimedes. Just as the theory of proportion of areas was made to depend essentially upon Pascal's Theorem, we will apply the same to develop the concept of area in our elementary geometry.

Hilbert first established the point that polygons could be broken into smaller polygons by drawing a line segment from one point on the polygon to another. The larger polygon is composed of the smaller ones, or was decomposed into them. Two polygons are said to be of equal area if we are able to decompose them into a finite number of triangles that are congruent to each other in pairs. He also defines the concept of equal content to be when polygons are not equal shape, but do enclose the same area. He states that if two polygons, P_1 and P_2, are each equal in area or content to a third polygon, P_3, then the first two have the same relationship to each other. Equality in area, and/or content, is transitive. He proved this by the simultaneous decomposition of polygons into a finite number of triangles.

He then defined, in the usual manner, the terms rectangle, base and height of a parallelogram, base and height of a triangle. He did this to illustrate the special case of this statement in which two parallelograms having equal bases and altitudes are also of equal content (see figure 9.1).

Similarly, two triangles having equal bases and heights have equal content, and a triangle contains equal area to a parallelogram that has an equal base and half the altitude of the triangle.

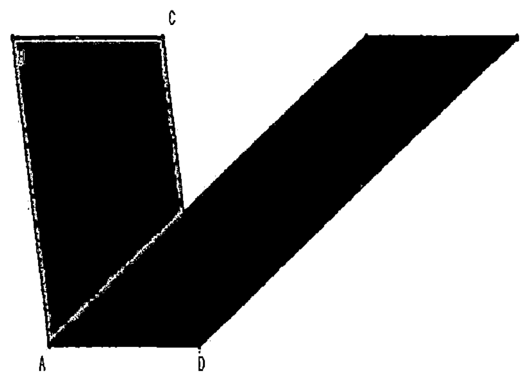

Figure 9.1: Triangles of Equal Content

The statement regarding the triangle and parallelogram is demonstrated in an accompanying illustration (see figure 9.2).

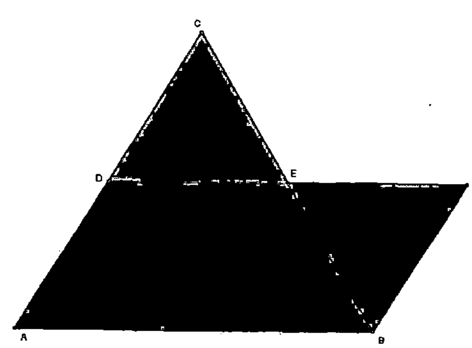

Figure 9.2: Triangle with Equal Area to Parallelogram

Sides AC and CB are both bisected. Segment DE is drawn to connect these midpoints. It is then extended an equal distance beyond E, to F. F is connected to B by another segment. As a result, triangles CDE and BFE are congruent to each other. Therefore, triangle ABC encloses an equal amount of area as does parallelogram $ABFD$.

Although it is common to show that two triangles with equal bases and altitudes contain equal area, it is of interest to note that this is dependent upon the axiom of

Archimedes. Without the axiom, we can only state that they have equal content.

I have attached a diagram of a shape that demonstrates triangles of equal base and altitude but a different shape (see figure 9.3).

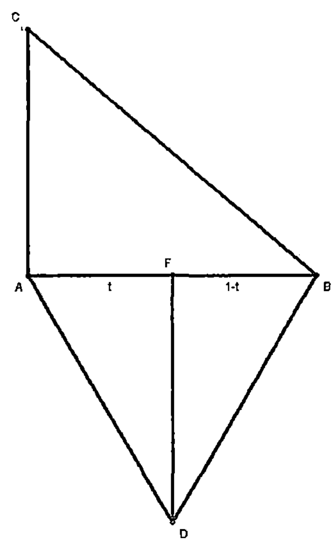

Figure 9.3: Equal Area versus Equal Content

ABC is a right triangle while ABD is not. With AB being a shared base, and the two altitudes (AC and DF) being equal in magnitude, the two triangles are equal in content.

The converse of the triangle statement is that if two triangles have equal content and equal bases, they also have equal altitudes. This fundamental theorem is included in Euclid's Elements as proposition 39. Euclid's theorem required introducing the concept of magnitude, essentially introducing a new geometrical axiom concerning area. Hilbert was able to accomplish the same task by using the plane axioms, even without the axiom of Archimedes. To do this, he needed to introduce the idea of the measure of area. This allowed him to present the topic without having to deal with the axiom of Archimedes.

If we construct two altitudes in triangle ABC (with sides of a, b, c) such that $h_a = AD$ and $h_b = BE$, then $\frac{a}{h_b} = \frac{b}{h_a}$ (see figure 9.4).

This is true due to the fact that triangles BCE and ACD are similar since they are right triangles that share a common second angle. By cross-multiplying these

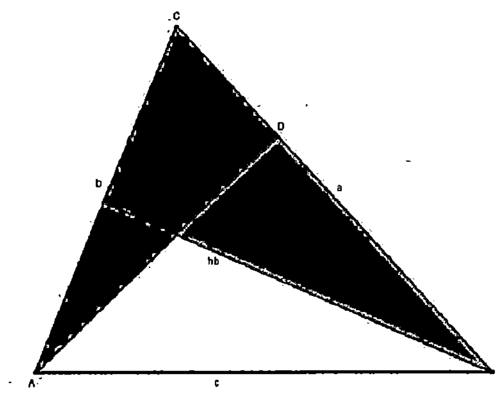

Figure 9.4: Area of Triangle

proportions, we get $a \times h_a = b \times h_b$. This shows that the product is the same regardless of which base and corresponding altitude is chosen. One-half of this product is called the measure of the triangle (Δ), and we denote it by $F(\Delta)$.

A segment joining a vertex of a triangle with a point on the opposite side is called a transversal. A transversal divides the given triangle into two others that have the same altitude and a base in the same line. This is called a transversal decomposition of the triangle.

A decomposition of a triangle need not be done through the use of transversals. If a triangle is decomposed by arbitrary straight lines into a finite number of triangles, then the area of the triangle is equal to the sum of measures of all the separate triangles. This is equivalent to repeated transversal decomposition on an arbitrary triangle. This same process can be used to decompose any polygon into triangles of equal measures of area. In so doing, we can demonstrate equality of area between polygons.

This process allows us to compare polygons of equal content and conclude that polygons of equal content are also of equal area.

The converse of the above statement is also true: polygons of equal area have equal content. In order to prove this, let us consider two triangles, ABC and $AB'C'$, that share a common right angle at A (see figure 9.5).

We know that the area of ABC is equal to half the base times the height. The area of the second triangle can be presented in a like manner and set equal to the first, since it is given. These facts give us the equation $\frac{1}{2}(AB)(AC) = \frac{1}{2}(AB')(AC')$.

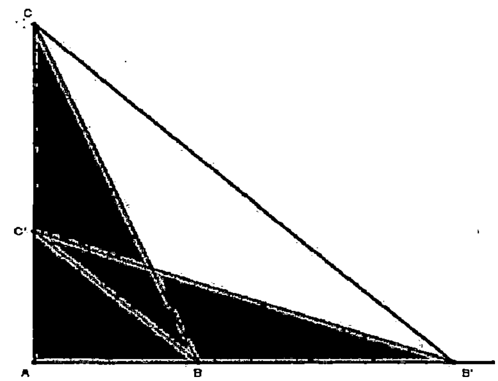

Figure 9.5: Equal Area means Equal Content

Therefore, $\frac{AB}{AB'} = \frac{AC'}{AC}$. From this statement, we know that BC' and $B'C$ are parallel, and that the triangles $BC'B'$ and $BC'C$ have equal content. By adding the common triangle ABC' to each of them, it follows that triangles ABC and $AB'C'$ have equal content.

The restriction to the right triangles was not necessary. A triangle having base and altitudes of g and h would have equal content to a right triangle whose legs measure g and h. Therefore, two arbitrary triangles with equal measures of area are also of equal content.

This concept is not limited to triangles, but can be generalized to include all polygons. This can be demonstrated by decomposing the polygons into triangles of equal content. Therefore, two polygons of equal content have equal measures of area, and two polygons of equal area have equal content. Therefore, if we decompose a rectangle into triangles and then omit one triangle, the resultant shape does not contain the same area as the original rectangle.

Chapter 10

Desargues's Theorem

When two triangles are situated in a plane in such a way that their corresponding sides are parallel, then the lines joining their matching vertices pass through a single point, or are parallel to one another. The converse is also true. If the triangles are situated such that the straight lines joining the corresponding vertices intersect in a common point or are parallel, and two sides of the triangles are parallel, the remaining sides are also parallel. This is Desargues's Theorem.

It has been said that this theorem is a consequence of the first three groups of axioms, and therefore, any geometry that accepts the plane axioms must also accept Desargues's theorem. However, Hilbert illustrated a geometry that followed the three sets of axioms, but in which the theorem in question failed. In order to make it true the axioms of congruence were required.

While he used the notation (x, y, z) to represent a point in space, he used $(u : v : w : r)$ to represent the plane the point is in. These variables represented the coefficients and constant in the standard form equation. It would be equivalent to writing an equation as a row vector. $(u : v : w : r)$ represents the plane $ux + vy + wz + r = 0$. He noted that plane geometry is merely a part of the geometry of space since it can be obtained by merely setting $z = 0$.

The geometry is constructed about the x and y axes and an ellipse about the origin. The ellipse intersects the major and minor axes at 1 and $\frac{1}{2}$, respectively, and is the graph of $x^2 + 4y^2 = 1$. It has foci at $(\sqrt{\frac{3}{2}}, 0)$ and $(-\sqrt{\frac{3}{2}}, 0)$. Let F denote a point on the positive x-axis $\frac{3}{2}$ units from the origin, outside the ellipse (see figure 10.1).

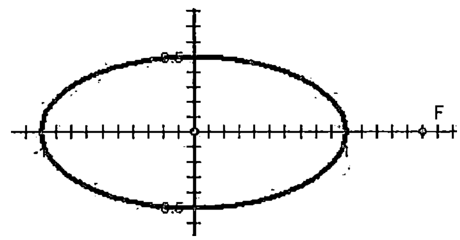

Figure 10.1: Hilbert's Geometry

In this geometry, any line that does not intersect the ellipse, or is merely tangent to it, remains unchanged. However, when a line crosses the ellipse in two points, such as P and Q, we will redefine the nature of the line (see figure 10.2).

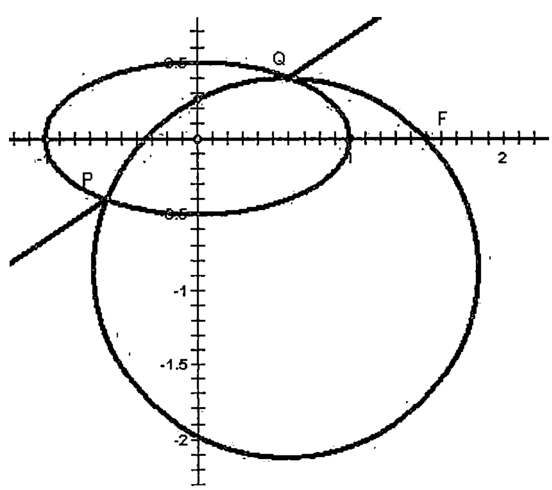

Figure 10.2: Redefining a Line

Construct a circle that passes through F, P and Q. The segments of the original line that lie outside the ellipse remain unchanged. However, the portion that lies inside, between P and Q, becomes PQ, the arc of the circle already drawn. In this new geometry, all the linear axioms hold. For example, two points determine a line. This is true since three points determine a circle, and the third point is F. The axioms of order, or betweenness, are unchanged. However, the axiom of parallel lines does not hold. For example, it is possible to have a line drawn through the ellipse that has no parallel line through a point chosen on the ellipse.

We will define two line segments, AB and $A'B'$, to be congruent if the broken line between the two endpoints are equal in length, in the ordinary sense of the word. We also need to be able to determine congruence of angles. When the angles are outside the ellipse, the ordinary definition holds. However, in other cases, ratios will be used to define congruence of angles. Let A, B, and C be points on a line in the new geometry (see figure 10.3).

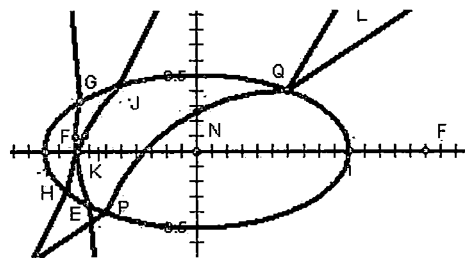

Figure 10.3: Congruence of Angles

Let A', B', and C' be points on another line in the geometry. Let D and D' be points not on these lines. Draw line segments BD and $B'D'$. Angle ABD is congruent to angle $A'B'D'$ if the ratio of the angles to their supplements are equal. In other words, $\frac{\angle ABD}{\angle CBD} = \frac{\angle A'B'D'}{\angle C'B'D'}$.

As stated earlier, in this geometry, Desargues's theorem does not hold. In order to see this, lets use the following three lines: the axes themselves and the line determined by the points $(\frac{3}{5}, \frac{2}{5})$ and $(\frac{-3}{5}, \frac{-2}{5})$ on the ellipse (see figure 10.4).

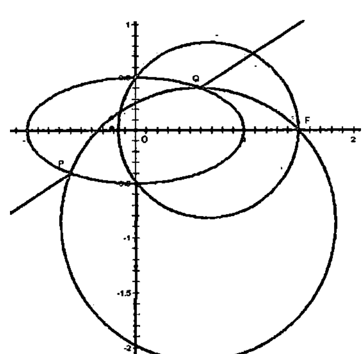

Figure 10.4: Failure of Desargues's Theorem

In Euclidean Geometry, these three lines intersect at the origin. Therefore, if we used the origin as our perspector we could easily create two triangles that have parallel corresponding sides. Each of the three lines will include a pair of corresponding vertices. However, in this new geometry, the lines do not intersect at a common point. Therefore, Desargues's theorem fails to hold.

To demonstrate the significance of Desarguess theorem, Dr. Hilbert then creates a new plane geometry in which the plane axioms of the first three groups hold independent of the axioms of congruence.

Now, define the length of a segment which starts at O to be represented by its other endpoint's small letter equivalent. For instance, the length of OA is a, while the lengths of OB and OC are represented by b and c, respectively.

Drawing two lines from O, create an angle (see figure 10.5).

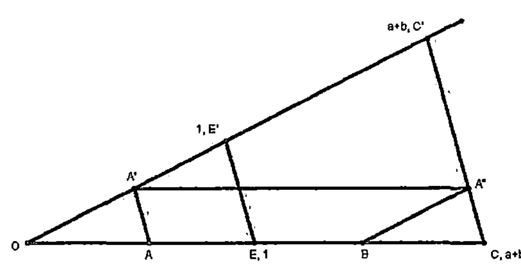

Figure 10.5: Geometry Independent of Axioms of Congruence

Let E and E' be drawn on the two lines in such a way that the lengths of OE and OE' are defined to be equal, and of length 1. Also, connect E to E' by use of the segment EE'.

Since this geometry is independent of the axioms of congruence, the lengths of OE and OE' need not be equal in a Euclidean sense. Equality is dependent on being parallel to the unit line. The segment that joins E to E' will be called our unit-line. If the Euclidean length on one side is b units longer for the original 'unit', that 'extra' portion is passed on to the segments created by the parallel lines. So this 'equality' is not one of ratios but of parallels to a predetermined relation. Also, the points that give equal lengths from O on the two angles are not necessarily unique.

Therefore, *by definition*, $OE = OE' = 1$ and the line segment EE' is the unit line. On OE lies the point A. Drawing a line parallel to the unit line, we intersect OE'. We name the point of intersection A'. On the line OE, but beyond E, designate a point B. Draw a line through B that is parallel to OE'. Now draw a line through A' that is parallel to OE. Name the intersecting point of these two lines A''. Now draw a line through A'' that is parallel to the unit line. Where this line intersects with OE and OE' will be named C and C', respectively.

In the figure above, AA', EE', and CC' are all parallel. In addition, $A'A''$ is parallel to OC. Now, by the use of parallel lines, it is easily seen that $AC = A'A'' = OB = b$. By subtracting AB from both sides of the $OB = AC$ equation, we find that $OA = BC$. Therefore, the length of OC (which, by definition, is c) is the sum of BC (whose length is now known to be a) and OB (defined to be b). Therefore, $a + b = c$. This diagram represents the addition of segments.

The product of segments is represented in the next diagram (figure 10.6).

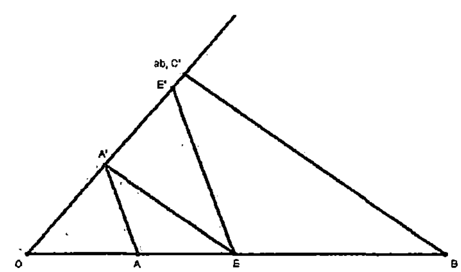

Figure 10.6: Product of Segments

In it, we again draw two lines from a single point, O. On the lines we will designate the points E and E', such that the segments OE and OE' will be one unit in length each, *by definition*. EE' will again be the unit line. Choose points A and A' on OE and OE', respectively, such that AA' is parallel to the unit line. Now connect E to A' with a line segment. We now choose a point B on the line OE (but past E). Draw a line through B that is parallel to $A'E$. Designate the point that intersects with OE' to be C'. Due to included angle at O and the parallel lines across from it (which force the other corresponding angles to be equal in measure), triangle $A'OE$ is similar to $C'OB$. Therefore we can use proportions to determine that $\frac{a}{1} = \frac{c}{b}$. By cross-multiplication, we get $c = ab$. Therefore, the length of OC', defined to be c, is also ab.

Next, the addition properties (Commutative and Associative) will be investigated in this new geometry. We again lay out two lines from the same point, O (see figure 10.7).

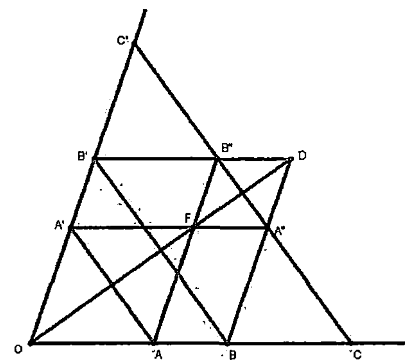

Figure 10.7: Commutative Law of Addition

On one line, we name points A and B. On the other, we name point A', which we will define to have length a (the same as the length of OA). We connect A and A', and designate AA' the unit line. We then draw a line parallel to AA', through B. Where this new line intersects with line OA', we call the point B'.

Next, we draw lines through A and B that are parallel to OB' and lines through A' and B' that are parallel to OB. These two sets of parallel lines intersect at four points. The diagonals of this newly formed quadrilateral are important. One will go through O. The points of intersection will be designated F and D, and the line is OFD. The other two points will be named B'' and A''. The line $B''A''$ will intersect OB' and OB at C' and C, respectively. Using the parallels, as we did before (in the addition of segments), we can show that $c - a = b$, and therefore $c = b + a$. Similarly, $c - b = a$ implies that $c = a + b$. Transitively, $a + b = b + a$. It is important to note that, in this new geometry, the commutative property only holds if $B''A''$ is parallel to AA'.

More importantly, in the diagram Desargues's theorem is illustrated using the triangles $AA'F$ and $BB'D$. Each of their corresponding sides is parallel and lines that connect the corresponding vertices meet at O.

In order to prove the associative law of addition, we will use another illustration (see figure 10.8),

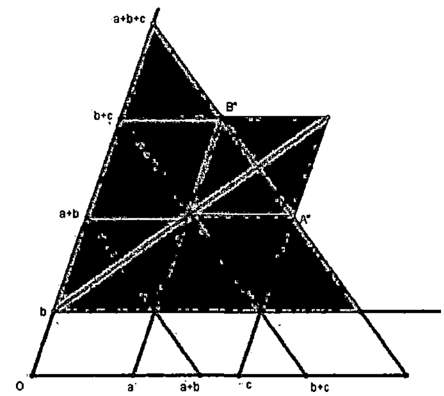

Figure 10.8: Associative Law of Addition

but we will still use the technique of parallelograms that we used for the commutative property earlier. $(a+b)+c$ is represented by the adjacent sides of a parallelogram with opposite vertices at O and A''. Similarly, $a+(b+c)$ is represented by a parallelogram with opposite vertices at O and B''. In order for the two lengths be equal, the line $B''A''$ must be parallel to the unit line. Since, the shaded part of this shape is the previous figure that we used, we know this to be so. Therefore, $(a+b)+c = a+(b+c)$.

The associative law of multiplication also has a place in our new algebra of segments (see figure 10.9).

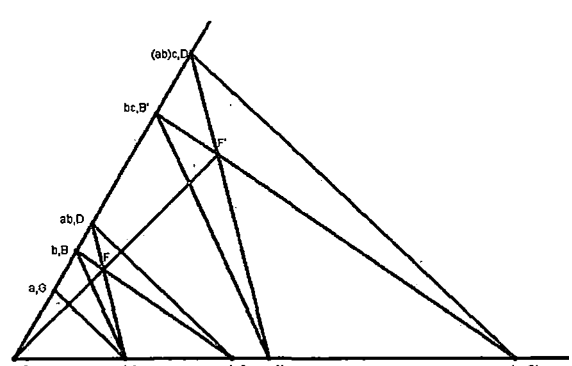

Figure 10.9: Associative Law of Multiplication

In the first of two new lines through O, let $1 = OA$, $b = OC$, and $c = OA'$ and on the second line let $a = OG$ and $b = OB$. Remember that, *in a Euclidean sense*, the lengths of OC and OB need not be the same. In this new geometry, they are *defined* as equal in length. With each segment designated as being of length b, we will call BC the unit line. Any additional lengths are determined to be the same based on being parallel to the unit line.

In order to prove this, we must construct the segments $bc = OB'$, $bc = OC'$, $ab = OD$ and $(ab)c = OD'$. We do this by drawing $A'B'$ parallel to AB. By similar triangles, we get $\frac{OB}{OA} = \frac{OB'}{OA'}$, which is equivalent to $\frac{b}{1} = \frac{OB'}{c}$. As a result, $OB' = bc$. By constructing BC parallel to $B'C'$, and using the same technique of ratios, we show that $bc = OC'$ also. We draw CD parallel to AG, and then we can construct $A'D'$ parallel to AD. By similar triangles, we can show that $OD = ab$ and $OD' = (ab)c$. Also, $\frac{OC'}{OA} = \frac{OD'}{OG}$ by similar triangles. Substituting in values, we get $\frac{bc}{1} = \frac{(ab)c}{a}$. By multiplying both sides by a, we get $a(bc) = (ab)c$, thus demonstrating the Associative property of multiplication.

Name the point of intersection of $A'D'$ and $B'C'$, F', and the junction between AD and BC, F. The resultant triangles ABF and $A'B'F'$ have their corresponding sides parallel to each other. According to Desargues's theorem, O, F, and F' must lie in a straight line. As a result of these conditions, we can apply the second part of his theorem to triangles CDF and $C'D'F'$ and show that CD is parallel to $C'D'$. Therefore, $\frac{OC'}{OC} = \frac{OD'}{OD}$, which is equivalent to saying $\frac{bc}{b} = \frac{(ab)c}{ab}$, which is true.

And, finally, based on Desargues's theorem, we will demonstrate that the two distributive laws, $a(b + c) = ab + bc$ and $(a + b)c = ac + bc$, also hold in this algebra of segments. In the proof of the first of these, we will make use of a new drawing. In figure 10.10, we designate the following lengths: $OA' = b$, $OC' = c$, $OB' = ab = OA''$, and $OC'' = ac$.

This proof is based on a proliferation of parallel lines. For example, $B''D_2$ is parallel to $C''D_1$ and OA', while $A'B''$ is parallel to $B'A''$, $F'D_2$, and $F''D_1$. It is also true that $A'A''$ is parallel $C'C''$. In order to prove this, we must show that $F'F''$ is also parallel to the last two segments mentioned. To demonstrate this, lines parallel to OA' and OA'' are drawn through F'' and F', respectively. The point of intersection between these two lines will be named J. The intersections created by $C''D_1$ and $C'D_2$, $C''D_1$ and $F'J$, and $C'D_2$ and $F''J$ will be denoted G, H_1, and H_2, respectively. All other lines

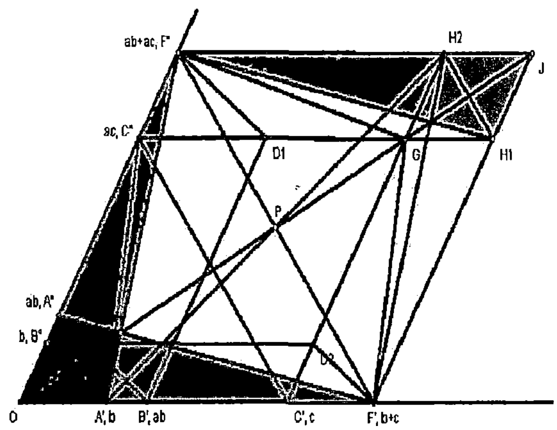

Figure 10.10: Distributive Laws

are constructed by conjoining points already determined.

The lines that connect the related vertices of triangles $A'B''C''$ and $F'D_2G$ are parallel to each other. The same is true for triangles $A'C''F''$ and $F'GH_2$. According to the second part of Desargues's theorem, each of the sides are parallel to its corresponding side on the other triangle. Therefore, $A'C''$ is parallel to $F'G$, as is $A'F''$ to $F'H_2$.

Since the homologous sides of triangles $OA'F''$ and JH_2F' are parallel, by Desargues's theorem, the segments connecting the related vertices will meet in a single point, P. As a result, by the second part of Desargues's theorem, $A''F'$ must be parallel to $F''H_1$.

In addition, since the homologous sides of $OA''F'$ and JH_1F'' are also parallel, the three straight lines joining their respective vertices (OJ, $A''H_1$, and $F'F''$) will also meet in a single point, namely P.

The lines connecting the homologous vertices of triangles $OA'A''$ and JH_2H_1 will also pass through P. Therefore, we know that H_1H_2 is parallel to $A'A''$, and necessarily also parallel to $C'C''$.

Finally, lets consider the figure $F''H_2C'C''H_1F'F''$. In this polygon, $F''H_2$ is parallel to both $C'F'$ and $C''H_1$. Also, H_2C' is parallel to both $F''C''$ and H_1F'. We also note that $C'C''$ is parallel to H_1H_2. This shape is the same one that we used to prove commutative property of addition (figure 10.11).

This confirms our earlier work, that requires $F'F''$ to be parallel to the last two

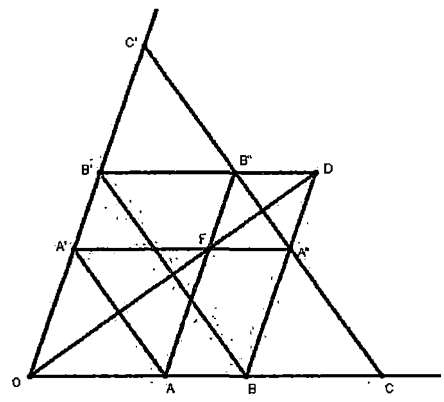

Figure 10.11: Commutative Property of Addition

line segments mentioned above ($C'C''$ and H_1H_2).

To more clearly show the first distributive property, observe the lengths of the line segments along OA and OA". The length of OA is b, while the length of OA'' is ab. The length of OC' is c, while the length of OC'' is ac. The parallel lines ($A'A''$, $C'C''$, and $F'F''$) require the ratio of the length on the OA'' side of the figure to be a times longer than the length of the segment on the OA' side. Consequently, OF'' must be a times longer than OF' (which is $b+c$). As a result $a(b+c) = ab + ac$.

The second statement of the distributive property, $(a+b)c = ac + ab$ requires a separate figure (see figure 10.12).

In this figure, we define the lengths of the following nine segments: OA, OD, OA', OD', OB, OG, OJ, OB', and OG'. The first four segments are on one line, and the remaining five are on another. The two lines are joined at O. The lengths are a, 1, ac, c, a, b, $a+b$, ac, bc, respectively. Furthermore, GH is parallel to $G'H'$ and the first fixed line OA, while AH is parallel to $A'H'$ and the second fixed line OB. In addition to this, AB is parallel to $A'B'$, BD is parallel to $B'D'$, DG is parallel to $D'G'$, and HJ is parallel to $H'J'$. What I must do is show that DJ is parallel to $D'J'$. The points where BD and GD intersect AH, I have named C and F, respectively. I have done the same with the points where $B'D'$ and $G'D'$ cross $A'H'$. Those points are called C' and F', respectively.

The homologous sides of ABC and $A'B'C'$ are parallel. Therefore, according

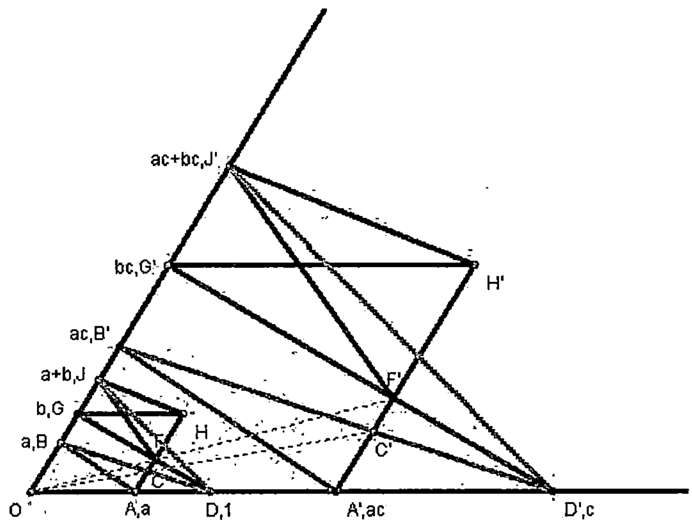

Figure 10.12: The Second Statement of Distributive Property

to Desarguess theorem, the three points O, C, C' lie in a straight line. Since the same situation occurs with triangles CDF and $CD'F'$, it follows that the points O, F, F' also lie in a straight line. Now, in the triangles FHJ and $F'H'J'$, the lines joining the respective vertices all pass through the same point, O. Therefore, according to the second part of Desargues's theorem, FJ and $F'J'$ must also be parallel to each other. Finally, a similar study of the triangles DFJ and $D'F'J'$ shows that lines DJ and $D'J'$ are also parallel to each other. As a result, the length of OJ', which is $ac+bc$ due to the parallelogram $OA'H'G'$, and the fact that $J'H'$ is parallel to JH, is also $(a+b)c$. The second statement is a result of $D'J'$ being parallel to DJ. Any lines parallel to it have a ratio of c. And therefore, the proof is complete, and $(a+b)c = ac + bc$.

The algebra of segments presented by Hilbert assumed that Desargues's theorem would hold. He used that to prove that Commutative Property of Addition, Associative Properties of Multiplication and Addition, and both Distributive laws held. And this was all done independent of the Axioms of Congruence. Now, we will show that an analytical representation of the point and straight line are possible using this algebra of segments as its basis.

Our co-ordinate axes, denoted x and y, will be any two fixed, straight lines in a plane that intersect. They do not need to be perpendicular to each other. We will designate the point of intersection as O. Any point P in the plane can now be represented by its co-ordinates, which are determined by drawing, through P, lines parallel to the two axes. Where those parallel lines intersect with the axes will determine the x and y values of the co-ordinates.

We will now deduce that the co-ordinates of a point, P, on any arbitrary line in the plane can be represented by an equation of these segments in the form $ax+by+c=0$. In this equation, a and b stands necessarily to the left of the co-ordinates x and y since the commutative property of multiplication is not proven in this algebra. The segment lengths (or values of) a and b are never both zero, and c is an arbitrary segment length (or value). Conversely, any equation that fits this description represents a straight line in this geometry.

The proof of this statement uses figure 10.13.

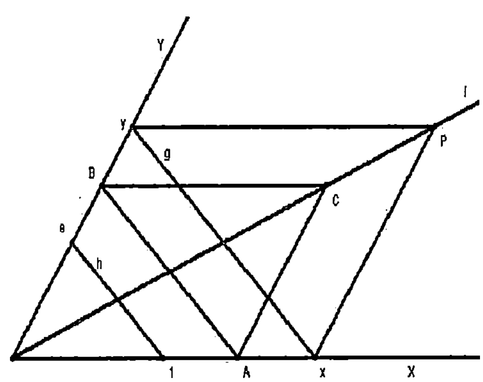

Figure 10.13: Straight Lines in Geometry

A line, l, is drawn through the origin, O. Points, C and P, are drawn on this line. The respective co-ordinates of these points are (OA, OB) and (x, y). The line segment joining x and y will be denoted g. A segment connecting A and B will also be drawn. Along x, the unit segment will be marked. A line parallel to AB, will be drawn through this unit segment. The point where this line intersects y, will be named e. The segment joining 1 to e will be called h. Since the lines connecting the corresponding vertices of triangles ABC and xyP all meet in one point, O, by Desargues's theorem, the corresponding sides are parallel. Therefore, g is parallel to AB, which is parallel to

h. In this algebra, when a line is parallel to another, the ratio of its extremities is the same. Since h creates a ratio of e, $y = ex$.

Now (in figure 10.14),

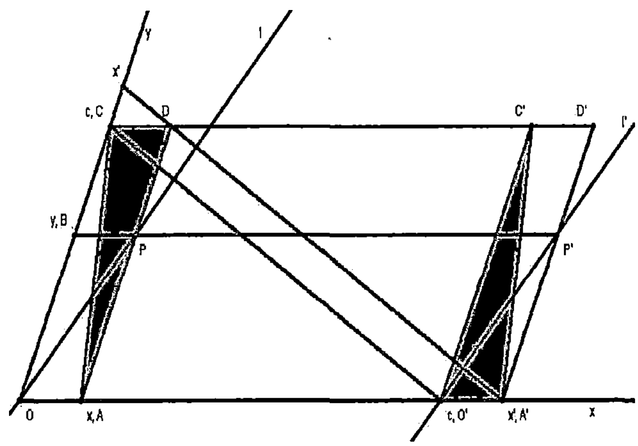

Figure 10.14: Parallel Lines Create Equal Proportions

we draw an arbitrary straight line, l', in the plane that intersects the x-axis at O'. The length of OO' will be called c. Then we draw a line, l, through the origin and parallel to l'. Let P' be an arbitrary point on l'. Draw lines through P' that are parallel to the axes. The intersections of these lines with the x and y axes will be denoted A' and B, respectively. The lengths of OA' and OB are x' and y, respectively. We will now show that $x' = x + c$ in this figure. As before, we will do this using a series of parallel lines. From O' draw a line parallel to the unit line, so that it cuts from the y-axis a segment of length c. Name the point of intersection C. From C, draw the line segment CD parallel to the x-axis. From D, draw a parallel to the y-axis. The point that intersects the x-axis will be named A, and the length of OA will be said to be x units. To show that $x' = x + c$, will amount to proving that $A'D$ is parallel to $O'C$. We will prove this through the use of Desargues's theorem.

The intersection of CD and $A'P'$ we will denote D'. Since the lines connecting the homologous vertices are parallel, by Desargues's theorem, CP is parallel to $C'P'$. Similarly, using triangles ACP and $A'C'P'$, AC must be parallel to $A'C'$. And, since the homologous sides of triangles ACD and $C'A'O'$ are parallel to each other, the straight lines AC', CA', and DO' intersect in a common point. As a result, the homologous sides of triangles $C'A'D$ and ACO' are parallel. One pair of such parallel sides is $A'D$ and CO'.

We have now shown that $y = ex$ and $x' = x + c$. By first solving for x in the second equation, and secondly multiplying both sides of that equation on the left by e, and finally replacing the ex by y,it follows that $ex' = y + ec$. As a result, $ex' - y - ec = 0$, and, as stated earlier, this is of the form $ax + by + c = 0$. To prove the converse, that any equation of the form $ax + by + c = 0$ represents a straight line in this geometry, we may multiply the left hand side of this equation by a well-chosen segment, to make it $ex - y - ec = 0$. It must be stated that the segment lengths (values), a and b, must be set to the left side of the equation. Since the Commutative Property of Multiplication was not proven in this algebra, one can not assume that $xa + by + c = 0$ represents a line in this geometry.

In this algebra of segments, the following properties hold: the operations of addition and multiplication, the existence of identities in these operations, and the fact that unique numbers are needed to add or subtract to reach a desired sum or product. However, the number needed to achieve the desired sum or product may be different when applied to the right or left side of the original term (since the Commutative Property of Multiplication has not been proven for this system). In addition to all this, by the use of Desargues's theorem, we have shown that the associative properties hold for these operations, the commutative property holds for addition, and both distributive properties apply. Therefore, with the single exception of the Commutative Property of Multiplication, all the theorems of connection hold.

Now, in order to compare the magnitude of segments, we will make a statement. Let A and B be two separate points on the straight line OE (no figure attached). If we view this line as if it were the number line with O being the zero value, and the segments OA and OB representing positive values and AO and BO representing negative values, we can compare the values. Larger negatives are smaller than smaller negatives and positives are larger than any negative. We let $a = OA$ and $b = OB$. Therefore, we can state that $a > b$ or $b > a$, depending on their placement on the line. The same can be done when $a = AO$ and $b = BO$. This convention holds true even if A or B coincide with O or E. This demonstrates the thirteenth statement in the theorems of connection. The next statement, $a > b$ and $b > c$ implies $a > c$ is easily seen through use of the segments.

Therefore, magnitude relationships are transitive. The fifteenth statement states that if $a > b$, then $a+c > b+c$ and $c+a > c+b$. So we can add equal lengths to two segments without affecting the inequality relationship. And, finally, the sixteenth statement holds that if $a > b$ and $c > 0$, then $ac > bc$ and $ca > cb$. Therefore, all the usual laws of operations hold except the Commutative Property of Multiplication and the Theorem of Archimedes. Such a system is called a desarguesian number system.

In this system, all the axioms of connection, order, and parallels hold. Hilbert represented a plane as a ratio of four numbers $(u : v : w : r)$, which we would be inclined to write as $ux+uy+wz+r = 0$. A line is defined to be an intersection of two non-parallel planes, and a point is represented by (x, y, z). A point is said to be part of a line if, and only if, it is contained in both the planes that define it. Collinear lines are not regarded as distinct.

All the axioms of connectedness and parallels (I and III) are satisfied by this geometry. In order for the axioms of order (II) to hold, we use the following convention. Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be any three points on the straight line designated by the intersection of the two planes $u'x+v'y+w'z+r' = 0$ and $u''x+v''y+w''z+r'' = 0$. This would have been presented by Hilbert as the line $[(u' : v' : w : r'), (u'' : v'' : w'' : r'')]$, the points determined to be the intersection of two planes. Of the three points, the second point would be said to be between the other two points if one of three statements was true. One possibility would be for the second x value to fall between the other two ($x_1 < x_2 < x_3$ or $x_1 > x_2 > x_3$). The other two possibilities involve the same inequalities involving the y or z components. If one of the inequalities involving x holds, it is easily seen, since the three points are all on the same straight line, that either the y components are equal or one of the compound inequalities also apply to the y values. The same holds for the z components.

This shows that the first four linear axioms (II), are all valid. Left to prove is fifth axiom of the second set, the one to do with planes. This axiom states that if A, B, and C are three points not lying on a straight line, and line a lies in the same plane and intersects segment AB (between A and B), it will also intersect segment BC or AC (see figure 10.15).

Hilbert calls the plane $(u : v : w : r)$ and the line in question $[(u : v : w : r), (u' : v' : w' : r')]$. The line is the intersection of the two planes listed in the brackets.

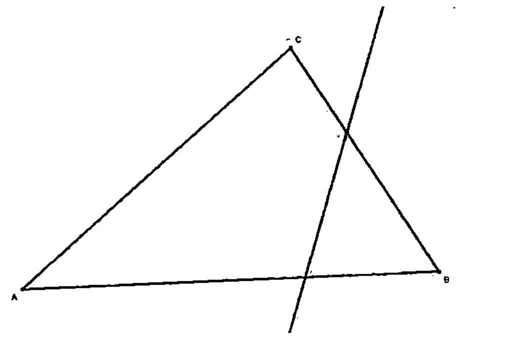

Figure 10.15: Axiom II-5 Revisited

Therefore, the plane, in modern notation, consists of all points that are solutions to the equation $ux + vy + wz + r = 0$. The line in question consists of all points that are solutions to that equation and $u'x + v'y + w'z + r' = 0$ (see figure 10.16).

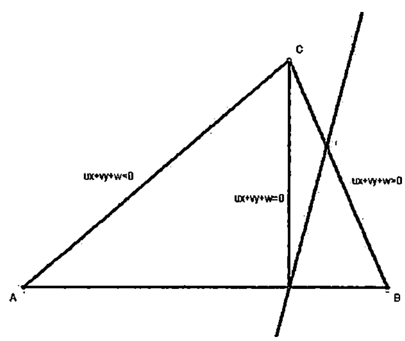

Figure 10.16: Solutions to Linear Equation

We will define the triangle and line in such a way that the line intersects segment AB and point C. Therefore, anytime the value of $u'x + v'y + w'z + r'$ is less than, or greater than, zero, it crosses one of the other two segments of the triangle - either AC or BC.

Therefore, all the axioms of connection, order, and parallels (groups I, II, and III) hold for this geometry of space using desarguesian numbers.

Earlier, Hilbert showed that the totality of all the different segments formed a complex number system. This now shows the converse of that proposition. The complex number system can be regarded as a geometry.

Chapter 11

Hilbert's Conclusion

In a plane geometry, if two points determine a line and the axioms of order and parallels hold, in addition to the theorem of Desargues, it is always possible to introduce into this geometry an algebra of segments in which all the normal operations and laws (commutative, associative, distributive) hold. The sum total of these segments can be used as the basis of a complex number system to represent the geometry of space. In this geometry of space, Hilbert considers only the points $(x, y, 0)$ and the straight lines that lie upon these points.

Hilbert concludes that in a plane geometry, if two points determine a line and the axioms of order and parallels hold, the existence of Desargues's theorem is both necessary and sufficient to fulfill the rest of the axioms of connection.

To quote Hilbert (in this English translation),

> The theorem of Desargues may be characterized for plane geometry as being, so to speak, the result of the elimination of the space axioms.

These results allow one to show that every geometry in which this holds can be presented as part of a geometry of any number of dimensions.

The series of lectures by Hilbert in the 1898-1899 school year (the basis of the *Foundations of Geometry* book) focused on the problems of Euclidean geometry. However, he notes in his conclusion that it is no less important to discuss the principles and fundamental theorems when the axiom of parallels is disregarded - the essence of non-Euclidean geometries.

Bibliography

[Hil59] Hilbert, David *The Foundations of Geometry* Open Court Publishing Company La Salle, Illinois 1959

[Rei70] Constance Reid *Hilbert* Springer-Verlag Berlin, New York 1970

[Gra00] Jeremy J. Gray *The Hilbert Challenge* Oxford University Press Oxford, New York 2000

[Rus56] Bertrand Russell *An Essay on the Foundations of Geometry* New York dover Publications New York 1956

[Ven06] Gerard A. Venema *Foundations of Geometry* Pearson Prentice Hall Upper Saddle River, New Jersey 2006

[Kat98] Victor J. Katz *A History of Mathematics, An Introduction* Addison-Wesley Reading, Massachusetts 1998

[Kay01] David C. Kay *College Geometry, A Discovery Approach* Addison Wesley Reading, Massachusetts 2001

[Art99] Benno Artmann *Euclid - The Creation of Mathematics* Springer New York 1999

[FG87] Judith Veronica Field and Jeremy J. Gray *The Geometrical Work of Girard Desargues* Springer-Verlag New York 1987

[Git75] Arthur Gittleman *History of Mathematics* Charles E. Merrill Publishing Company columbus, Ohio 1975

[Sma73] James R. Smart *Modern Geometries* Brooks/Cole Publishing Company Monterey, California 1973